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On scattering systems related to the $SO(2,1)$ group

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Abstract. Scattering systems related to the noncompact groups G in the sense that the Hamiltonian of the system can be written as a function of the Casimir operator of G are considered. The S-matrix for such systems are defined in terms of an intertwining operator of underling symmetry group G. The S-matrices for one-dimensional scattering systems with $SO(2,1)$ symmetry group are classified.

1. Introduction

Integrable models provide the key to our understanding of more realistic interactions. They appear in different areas of physics both in classic and quantum domains. The group theoretical methods give a unified approach to a class of integrable systems related to Lie groups [1]. In the quantum case, the Hamiltonian H of the systems is expressed in terms of the Casimir operator C of symmetry group G, i.e. $H = f(C)$. Hence, this connection allows one to find the wavefunctions, spectra and S-matrices, without a direct solution of the Schrödinger equation. In this description, as in the conventional approach, the S-matrix is defined through the asymptotic behaviour of the scattering wavefunctions. Hence, the natural question arises as to whether S-matrices can be calculated algebraically. The beginning of such a program was presented in [2], where the authors suggested the construction of an algebraic framework to calculate the S-matrix for the Pöschl–Teller potential. However, the method employed there used coordinate realization. Subsequently, Alhassid and co-workers [3, 4] proposed a purely algebraic description of the S-matrix for scattering problems with $SO(2,1)$ dynamical symmetry (for a generalization to any number of dimensions see [5–7]). The recurrence relations for the S-matrix are obtained by writing the infinitesimal operators of representations of the dynamical group G in terms of those of asymptotic group G^0 describing the problem in the absence of interactions. However, since a general procedure for the description of such connection formulae is absent, it is not so easy to find the explicit form of the S-matrix with this method.

In a recent paper by one of the present authors [8] a new approach was initiated for such scattering systems. In that paper the S-matrices for systems under consideration are related to intertwining operators between Weyl equivalent principal series representations of the dynamical group G. In other words, the S-matrix for systems under consideration is constrained to satisfy the equation

$$SU^{\chi}(g) = U^{\tilde{\chi}}(g)S \qquad \text{for all } g \in G \tag{1.1}$$

or

$$SU^{\chi}(b) = U^{\tilde{\chi}}(b)S \qquad \text{for all } b \in \boldsymbol{g} \tag{1.2}$$

where U^{χ} and $U^{\tilde{\chi}}$ are the Weyl equivalent principal series representation of G while $U^{\chi}(b)$ and $U^{\tilde{\chi}}(b)$ are the corresponding representations of the algebra $\boldsymbol{g}$ of G. (The representations U^{χ} and $U^{\tilde{\chi}}$ have the same Casimir eigenvalues. Such representations are called Weyl equivalent.) Equation (1.1) or (1.2) is actually used in deriving the S-matrix.

At this stage we note that the operator S from H^{χ} to $\mathsf{H}^{\tilde{\chi}}$ is said to be intertwining if relation (1.1) or (1.2) hold, where $\mathsf{H}^{\chi}(\mathsf{H}^{\tilde{\chi}})$ is the carry space of the representation $U^{\chi}(U^{\tilde{\chi}})$ of G [9]. We shall see how one could in principle evaluate the S-matrix from (1.1) or (1.2) without ever writing a Schrödinger equation, or wavefunctions, or ever mentioning the concepts of space and time. Moreover, this method has led to the hope that one may be able to classify and may be determine explicitly all the S-matrices for systems with symmetry group. For simplicity we shall now restrict ourselves to a scattering problem related to the $SO(2,1)$ symmetry group.

2. Calculation S-matrices for $SO(2,1)$ group

Let the scattering systems be related to the noncompact group $SO(2,1)$ in the sense that the Hamiltonian of the system can be written as a function of the Casimir operator of $SO(2,1)$. Then, the S-matrices for such systems can be defined from equation (1.1) or (1.2). To this end, a few facts from representation theory of the group $SO(2,1)$ are useful.

The unitary irreducible representations (UIRs) of $SO(2,1) \approx SU(1,1)$ are known [9] to form three series: principal, supplementary and discrete. It is also known that (see, e.g. [10] and references to earlier work cited therein) only the principal series of the UIR of $SO(2,1)$ (the dynamical symmetry group) goes over in the Inönü–Wigner contraction limit into the UIR of $E(2)$ (an asymptotic symmetry group). Consequently, the relevant unitary representations will be the principal series and we restrict the discussion to it.

The principal series of $SU(1,1)$ are characterized by the pair $\chi = (\rho, \varepsilon)$, where ε is equal to 0 or $\frac{1}{2}$, while $0 \leqslant \rho < \infty$. The representations specified by labels $\chi = (\rho, \varepsilon)$ and $\chi = (-\rho, \varepsilon)$ are equivalent. The operators of the representation of the Lie algebra of $SU(1,1)$ associated with the principal series are denoted by J_i^{χ}, $i = 1, 2, 3$. J_i^{χ} are the Hermitian operators and satisfy the commutation relations

$$[J_1^{\chi}, J_2^{\chi}] = -\mathrm{i}J_3^{\chi} \qquad [J_2^{\chi}, J_3^{\chi}] = \mathrm{i}J_1^{\chi} \qquad [J_3^{\chi}, J_1^{\chi}] = \mathrm{i}J_2^{\chi} \tag{2.1}$$

where J_3^{χ} is elliptic and J_1^{χ}, J_2^{χ} are hyperbolic. The Casimir operator

$$C = -(J_1^{\chi})^2 - (J_2^{\chi})^2 + (J_3^{\chi})^2 \tag{2.2}$$

is identically a multiple of the unit $C = -\frac{1}{4} - \rho^2$.

We take as a scattering basis of the carry space the eigenvector $|m\rangle$ of J_3^{χ}, where $m = n + \varepsilon$, $n = 0, \pm 1, \pm 2, \ldots$. We introduce the following operators

$$J_{\pm}^{\chi} = \mathrm{i}J_1^{\chi} \mp J_2^{\chi}. \tag{2.3}$$

The operators J_+^{χ}, J_-^{χ}, J_3^{χ} act on the basis vectors in the following way [9]

$$J_3^{\chi}|m\rangle = m|m\rangle \tag{2.4}$$

$$J_{\pm}^{\chi}|m\rangle = (\tfrac{1}{2} - \mathrm{i}\rho \pm m)|m \pm 1\rangle. \tag{2.5}$$

Let us now show that equation (1.2) is sufficient to compute the S-matrix. To do this, let us write equation (1.2) explicitly

$$SJ_3^{\chi} = J_3^{\tilde{\chi}}S \tag{2.6}$$

$$SJ_+^{\chi} = J_+^{\tilde{\chi}}S \tag{2.7}$$

$$SJ_-^{\chi} = J_-^{\tilde{\chi}}S. \tag{2.8}$$

Applying both sides of equation (2.6) to the basis vector $|m\rangle$ we get

$$mS|m\rangle = J_3^{\tilde{\chi}} S|m\rangle. \tag{2.9}$$

Thus, the matrix of operator S in this basis is diagonal

$$\langle m'|S|m\rangle = S_m \delta_{m'm}. \tag{2.10}$$

The value of its diagonal elements can be defined from equation (2.7) or (2.8). Apply, for example, both sides of equality (2.7) to the basis vector $|m\rangle$. As a result we obtain the recurrence relation

$$(\tfrac{1}{2} - \mathrm{i}\rho + m)S_{m+1} = (\tfrac{1}{2} + \mathrm{i}\rho + m)S_m \tag{2.11}$$

which implies that

$$S_m = c(\rho)\frac{\Gamma(\frac{1}{2} + \mathrm{i}\rho + m)}{\Gamma(\frac{1}{2} - \mathrm{i}\rho + m)} \tag{2.12}$$

where $c(\rho)$ is a constant with modulus$= 1$. The energy-dependent parameter ρ is determined by the relation between the Hamiltonian H and the Casimir invariant C.

The Coulomb problem in two dimensions with Hamiltonian

$$H = \frac{1}{2}p^i p_i + \frac{\alpha}{\sqrt{x^i x_i}} \tag{2.13}$$

where p_i and x_i, $i = 1, 2$, are the linear momentum and coordinates, provides an example of a quantum system with the symmetry group $SO(2, 1)$ [11]. On a subspace spanned by eigenvector of H, the infinitesimal operators J_i^{χ}, $i = 1, 2, 3$ are defined by

$$J_i^{\chi} = (2H)^{-1/2}A_i \qquad i = 1, 2 \tag{2.14}$$

$$J_3^{\chi} = M \tag{2.15}$$

where A_i and M are the Runge–Lenze vector and momentum, respectively

$$\begin{aligned} A_1 &= \frac{1}{2}(-Mp_2 - p_2M) - \frac{x_1}{\sqrt{x_1^2 + x_2^2}} \\ A_1 &= \frac{1}{2}(Mp_1 + p_1M) - \frac{x_2}{\sqrt{x_1^2 + x_2^2}} \\ M &= x_1p_2 - x_2p_1. \end{aligned} \tag{2.16}$$

The relation between the Hamiltonian H and the Casimir operator C is given by

$$H = -\frac{\alpha^2}{2(C + \frac{1}{4})}. \tag{2.17}$$

Since $H = k^2/2$ and $C = -\frac{1}{4} - \rho^2$ it is clear that, for the Coulomb problem, $\rho = \alpha/k$. Hence formula (2.12) with $\rho = \alpha/k$ is determined as the S-matrix for the two-dimensional Coulomb problem.

Observe that, the operator S (see equation (2.10)) does not mix states belonging to different one-dimensional subspaces $\mathbf{H}_m$ spanned by $|m\rangle$. This fact leads to the suggestion that there might exist a class of one-dimensional potentials for which the S-matrix is determined by numbers S_m. This is, in fact, exactly what happens in the 'potential group' approach to scattering problems [12] (see also [2–7]) where the representations of group G describe states with the same energy but different potential strengths.

Moreover, we can extract corresponding one-dimensional potentials from the Casimir operator. To do this, let us consider, for example, a (reducible!) representation $T(g)$ of $SO(2,1)$ realized in the Hilbert space of square-integrable function $f(\xi)$ on an upper sheet of hyperboloid [13]

$$\xi_0^2 - \xi_1^2 - \xi_2^2 = 1 \qquad \xi_0 > 0 \tag{2.18}$$

with an invariant measure

$$\mathrm{d}\xi = \mathrm{d}\xi_1 \, \mathrm{d}\xi_2/\xi_0. \tag{2.19}$$

The representation $T(g)$ is defined by

$$T(g)f(\xi) = f(\xi g). \tag{2.20}$$

The infinitesimal operators J_i of this representation are given by

$$J_k = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}T(g_k(t))|_{t=0} \qquad k = 1, 2, 3 \tag{2.21}$$

where $g_1(t)$, $g_2(t)$ are the pure Lorentz transformations along the 1 and 2 axes, respectively, while $g_3(t)$ is rotations in the 1–2 plane. Hence,

$$J_1 = \mathrm{i}\xi_0\frac{\partial}{\partial\xi_1} \qquad J_2 = \mathrm{i}\xi_0\frac{\partial}{\partial\xi_2} \qquad J_3 = \mathrm{i}\left(\xi_2\frac{\partial}{\partial\xi_1} - \xi_1\frac{\partial}{\partial\xi_2}\right). \tag{2.22}$$

Now, we require the representation space to be irreducible. (We note that representation (2.20) is decomposed onto the direct integral of principal series representations $(\rho, 0)$ [13].) Such a restriction is obtained if all functions are eigenfunctions of the Casimir operator,

$$C = \frac{\partial^2}{\partial\xi_1^2} + \frac{\partial^2}{\partial\xi_2^2} + \left(\xi_1\frac{\partial}{\partial\xi_1} + \xi_2\frac{\partial}{\partial\xi_2}\right)^2 + \xi_1\frac{\partial}{\partial\xi_1} + \xi_2\frac{\partial}{\partial\xi_2} \tag{2.23}$$

of the Lie algebra (2.22), i.e.

$$Cf = (-\rho^2 - \tfrac{1}{4})f. \tag{2.24}$$

Generally we may choose a large number of different coordinate systems on the hyperboloid. The different choices of coordinate systems on the hyperboloid lead to different reductions of the group $SO(2,1)$ to its subgroup. The $|m\rangle$ basis is given by the decomposition according to the compact subgroup $SO(2,1) \supset SO(2)$. As a prelude to this decomposition one introduces the spherical coordinates on the hyperboloid (2.18) given by

$$\begin{aligned} \xi_0 &= \cosh\alpha \\ \xi_1 &= \sinh\alpha\cos\varphi \\ \xi_2 &= \sinh\alpha\sin\varphi. \end{aligned} \tag{2.25}$$

With the introduction of spherical coordinates and substitution of the function $f(\alpha, \varphi)$ by $\omega^{-1/2}\Psi(\alpha)\mathrm{e}^{\mathrm{i}m\varphi}$, where $\omega = \sinh\alpha$ is the weight function in the hyperboloid measure $\mathrm{d}\xi = \sinh\alpha \, \mathrm{d}\alpha \, \mathrm{d}\varphi$, the Casimir eigenvalue equation reduces to the Schrödinger equation

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2} + \frac{m^2 - \frac{1}{4}}{\sinh^2\alpha}\right)\Psi(\alpha) = E\Psi(\alpha) \tag{2.26}$$

with

$$E = \rho^2. \tag{2.27}$$

The Hamiltonian is now given by

$$H = -(C + \tfrac{1}{4}) \tag{2.28}$$

(on one-dimensional subspace $\mathbf{H}_m$). Thus, the knowledge of the intertwining operator in the $SO(2)$ basis solves the scattering problem for the Pöschl–Teller potential $V(\alpha) = (m^2 - \frac{1}{4})/\sinh^2\alpha$,

$$S_m = \frac{\Gamma(1 - \mathrm{i}\rho)\Gamma(\frac{1}{2} + \mathrm{i}\rho + m)}{\Gamma(1 + \mathrm{i}\rho)\Gamma(\frac{1}{2} - \mathrm{i}\rho + m)} \tag{2.29}$$

with $\rho = \sqrt{E}$ following from (2.27) or (2.28). Since the Pöschl–Teller potential with $m = \frac{1}{2}$ corresponds to the free case, we have chosen the phase factor c in (2.12) as

$$c(\rho) = \frac{\Gamma(1 - \mathrm{i}\rho)}{\Gamma(1 + \mathrm{i}\rho)}. \tag{2.30}$$

There are, however, a class of one-dimensional scattering systems related to $SO(2, 1)$ group which are not in the same above classes in the sense that their S-matrices differ from (2.12). In order to complete the program to find the S-matrices of problems with the $SO(2, 1)$ symmetry group, we have to calculate intertwining operators in all subgroup bases. We find it expedient to use, for this purpose, equation (1.1).

As is well known, the group $SO(2, 1)$ has three subgroups $SO(2)$, $SO(1, 1)$ and $E(1)$ generated by J_3, J_1 and $N = J_2 + J_3$, respectively. Hence, we are interested in examining the intertwining operator in $SO(1, 1)$ and $E(1)$ bases in which the operators J_1 and N are diagonal, respectively.

The basis vectors will be denoted in the usual fashion by the kets

$$J_1|\mu\tau\rangle = \mu|\mu\tau\rangle \qquad \langle\mu'\tau'|\mu\tau\rangle = \delta_{\tau\tau'}\delta(\mu - \mu') \tag{2.31}$$

with $-\infty < \mu < \infty, \tau = \pm 1$,

$$N|\lambda\rangle = \lambda|\lambda\rangle \qquad \langle\lambda'|\lambda\rangle = \delta(\lambda - \lambda') \tag{2.32}$$

with $-\infty < \lambda < \infty$. Note that each UIR of $SO(1, 1)$ is doubly degenerate in principal series of UIR of $SO(2, 1)$ and τ is the multiplicity label.

Now let us return to equation (1.1). By realizing the principal series of $SO(2, 1)$ on suitable Hilbert spaces of some functions we can derive from equation (1.1) the functional relations for the kernel of S which allow us to obtain an integral representation for the S-matrix. Thus, we can calculate S-matrix in a straightforward manner from its integral formula. Without going into calculational details, we simply list the results (see the appendix).

(a) In the $SO(1, 1)$ basis:

$$\langle\mu'\tau'|S|\mu\tau\rangle = \delta(\mu - \mu')S_{\tau\tau'}(\mu) \tag{2.33}$$

where

$$\begin{aligned} S_{++}(\mu) = S_{--}(\mu) &= \frac{c}{\pi}\cosh\pi\rho\,\Gamma(\tfrac{1}{2} + \mathrm{i}\rho + \mathrm{i}\mu)\Gamma(\tfrac{1}{2} + \mathrm{i}\rho - \mathrm{i}\mu) \\ S_{+-}(\mu) = S_{-+}(\mu) &= -\mathrm{i}\frac{c}{\pi}\sinh\pi\rho\,\Gamma(\tfrac{1}{2} + \mathrm{i}\rho + \mathrm{i}\mu)\Gamma(\tfrac{1}{2} + \mathrm{i}\rho - \mathrm{i}\mu). \end{aligned} \tag{2.34}$$

(b) In the $E(1)$ basis:

$$\langle\lambda'|S|\lambda\rangle = \delta(\lambda - \lambda')S_\lambda$$

where

$$S_\lambda = c\lambda^{2\mathrm{i}\rho}.$$

Thus, we have come to a very important conclusion; there exist three classes of one-dimensional scattering systems related to the $SO(2,1)$ group with S-matrices given by the following.

(i) Class 1 (related to reduction $SO(2,1) \supset SO(2)$)

$$S_m = \begin{pmatrix} R_m & 0 \\ 0 & R_m \end{pmatrix} \tag{2.35}$$

where

$$R_m = c(\rho)\frac{\Gamma(\frac{1}{2} + \mathrm{i}\rho + m)}{\Gamma(\frac{1}{2} - \mathrm{i}\rho + m)}. \tag{2.36}$$

(ii) Class 2 (related to reduction $SO(2,1) \supset SO(1,1)$)

$$S_\mu = \begin{pmatrix} R_\mu & T_\mu \\ T_\mu & R_\mu \end{pmatrix} \tag{2.37}$$

where

$$\begin{aligned} R_\mu &= c(\rho)\cosh\pi\mu\,\Gamma(\tfrac{1}{2} + \mathrm{i}\rho + \mathrm{i}\mu)\Gamma(\tfrac{1}{2} + \mathrm{i}\rho - \mathrm{i}\mu) \\ T_\mu &= -\mathrm{i}c(\rho)\frac{1}{\pi}\sinh\pi\rho\,\Gamma(\tfrac{1}{2} + \mathrm{i}\rho + \mathrm{i}\mu)\Gamma(\tfrac{1}{2} + \mathrm{i}\rho - \mathrm{i}\mu). \end{aligned} \tag{2.38}$$

(iii) Class 3 (related to reduction $SO(2,1) \supset E(1)$)

$$S_\lambda = \begin{pmatrix} R_\lambda & 0 \\ 0 & R_\lambda \end{pmatrix} \tag{2.39}$$

where

$$R = c(\rho)\lambda^{2\mathrm{i}\rho}. \tag{2.40}$$

It should be noted that the potential functions $V(x)$ of the second class admit a double degeneracy of the wavefunction for every positive value of E (see equation (2.31)). The situation here is analogous to that of the case of a square potential barrier [14]. The double degeneracy corresponds to the fact that one may construct wavepackets which are partly transmitted and partly reflected by the potential $V(x)$. According to (2.37) and (2.38), the reflection and transmission coefficients are

$$|R_\mu|^2 = \frac{\cosh^2\pi\mu}{\cosh^2\pi\mu + \sinh^2\pi\rho} \tag{2.41}$$

$$|T_\mu|^2 = \frac{\sinh^2\pi\rho}{\cosh^2\pi\mu + \sinh^2\pi\rho} \tag{2.42}$$

respectively. It is also worth noting that, according to (2.36) and (2.40), the reflection coefficient $|R_m|^2 = |R_\lambda|^2 = 1$ for all potentials of class I or II; hence the reflection is total. This is a result of very general properties, shared by all one-dimensional Hamiltonians which have continuous nondegenerate spectrum.

We conclude this section by extracting one-dimensional potentials from the Casimir operator of the representation (2.20) of $SO(2,1)$ corresponding to reductions $SO(2,1) \supset SO(1,1)$ and $SO(2,1) \supset E(1)$. According to this, one has to choose the following coordinate systems on hyperbola.

(a) Hyperbolic

$$\begin{aligned} \xi_0 &= \cosh\beta\cosh\alpha \\ \xi_1 &= \cosh\beta\sinh\alpha \\ \xi_2 &= \sinh\beta \end{aligned} \tag{2.43}$$

with $-\infty < \beta < \infty, -\infty < \alpha < \infty$.

(b) Parabolic (or horispherical)

$$\begin{aligned}\xi_0 &= \cosh\theta + \frac{x^2}{2}\mathrm{e}^\theta \\ \xi_1 &= \sinh\theta - \frac{x^2}{2}\mathrm{e}^\theta \\ \xi_2 &= x\mathrm{e}^\theta\end{aligned} \tag{2.44}$$

with $-\infty < \theta < \infty, -\infty < x < \infty$. The invariant measure on the hyperboloid in these coordinate systems are

$$\mathrm{d}\xi = \cosh\beta\,\mathrm{d}\beta\,\mathrm{d}\alpha \tag{2.45}$$

and

$$\mathrm{d}\xi = \mathrm{e}^\theta\,\mathrm{d}\theta\,\mathrm{d}x \tag{2.46}$$

respectively.

By arguments very similar to those used to obtain (2.26) we can show that [15, 16]

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}\beta^2} + \frac{\mu^2 + \frac{1}{4}}{\cosh^2\beta} \qquad \text{for } SO(2,1) \supset SO(1,1) \tag{2.47}$$

and

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + \frac{\lambda^2}{\mathrm{e}^{2\theta}} \qquad \text{for } SO(2,1) \supset E(1) \tag{2.48}$$

respectively. In both cases, the group Hamiltonians H are related to the Casimir invariant C by $H = -(C + \frac{1}{2})$. Therefore, formulae (2.37) and (2.39) at $\rho = \sqrt{E}$ determine the scattering matrices for the potentials $V = (\mu^2 + \frac{1}{4})/\cosh^2\beta$ and $V = \lambda^2/\mathrm{e}^{2\theta}$ presented in figures 1 and 2, respectively. Besides the above-mentioned applications, it is expected that using the other realizations of the representation of $SO(2,1)$ it will be possible to construct a family of new potentials.

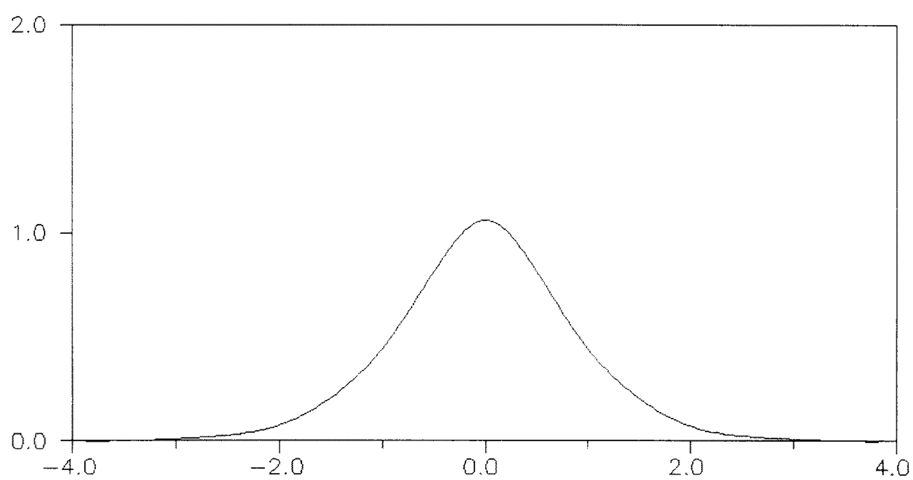

Figure 1. The potential function $V(\beta) = (\mu^2 + \frac{1}{4})/\cosh^2\beta$ is plotted for $\mu = 0.9$. The axes are in arbitrary units.

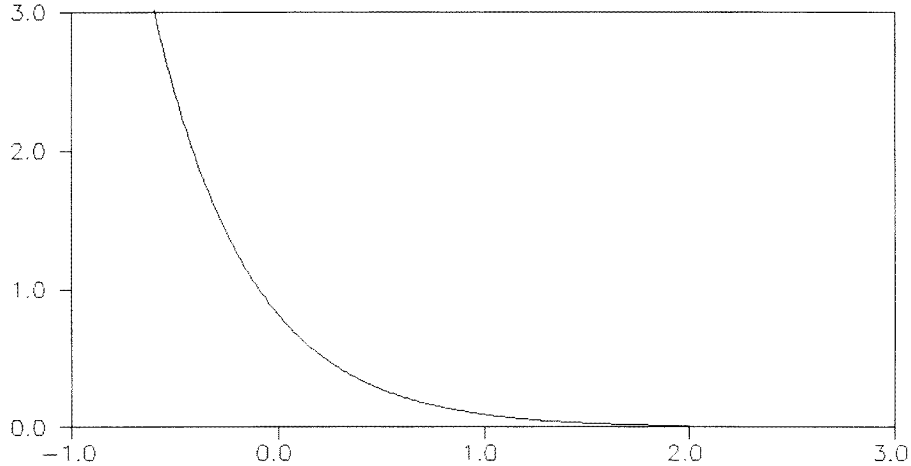

Figure 2. The potential function $V(\theta) = \lambda^2/\exp(2\theta)$ is plotted for $\lambda = 0.9$ The axes are in arbitrary units.

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Appendix

In this section we calculate intertwining operators of the group $SO(2,1)$ for all subgroup bases.

In order to fix notation and terminology we start with a brief description of elementary (or nonunitary principal series) representations U^j, $j \in \mathbf{C}$ of the group $SO(2,1)$. The representations U^j, can be realized in the space of infinitely differentiable function $F(\zeta)$ on the upper sheet of the two-dimensional cone $\zeta_0^2 - \zeta_1^2 - \zeta_2^2 = 0$, $\zeta_0 > 0$, homogeneous of degree j [13]

$$F(a\zeta) = a^j F(\zeta) \qquad a > 0. \tag{A.1}$$

The representations U^j are given by

$$U^j(g)F(\zeta) = F(\zeta g). \tag{A.2}$$

Note that we consider $SO(2,1)$ as acting on three-dimensional pseudo-Euclidean space $R^{1,2}$ with bilinear form $[\zeta, \eta] = \zeta_0\eta_0 - \zeta_1\eta_1 - \zeta_2\eta_2$ on the right. In accordance with this we shall write the vector in the row form $\zeta = (\zeta_0, \zeta_1, \zeta_2)$.

As mentioned, the different choices of coordinate systems on the cone lead to different subgroup reductions of $SO(2,1)$.

(i) The spherical coordinate system corresponding to the subgroup reduction $SO(2,1) \supset SO(2)$ is given by

$$\zeta = \omega n \qquad n = (1, \cos\varphi, \sin\varphi) \tag{A.3}$$

where $0 \leqslant \omega < \infty$, $0 < \varphi < 2\pi$.

From (A.1) it follows that the homogeneous function is defined uniquely by its values on the circle $S^1 \in n = (1, \cos\varphi, \sin\varphi)$. Consequently elementary representations of $SO(2,1)$ can be realized on the space C^∞ of infinitely differentiable functions $f(n) \equiv F(1, \cos\varphi, \sin\varphi)$ on S^1

$$U^j(g) f(n) = \left(\frac{\omega_g}{\omega}\right)^j f(n_g) \tag{A.4}$$

where ω_g and n_g are determined from the parametrization (A.3) of $\zeta_g = \zeta g$. Representations (A.4) with $j = -\frac{1}{2} + \mathrm{i}\rho$ can be extended (by an appropriate completion of C^∞) to principal series $(\rho, 0)$ of $SO(2,1)$ on the Hilbert space $L^2(S^1)$ with inner product

$$(f, f') = \frac{1}{2\pi} \int_0^{2\pi} \overline{f(\varphi)} f'(\varphi)\, \mathrm{d}\varphi.$$

The representations labelled by j and $-1-j$ are equivalent.

By virtue of the theorem on kernel, the operator S can be defined as

$$(Sf)(n) = \int_{S^1} K(n, n') f(n')\, \mathrm{d}n' \tag{A.5}$$

where $\mathrm{d}n \equiv \mathrm{d}\varphi$ is the invariant measure on S^1. Thus, equation (1.1) will serve to fix the dependence of the kernel $K(n, n')$ on n and n'. Equality (1.1) implies that

$$(SU^j(g) f)(n) = (U^{-1-j}(g) Sf)(n). \tag{A.6}$$

So, the kernel $K(n, n')$ is constrained to satisfy the functional equation

$$K(n_g, n'_g) = \left(\frac{\omega_g}{\omega}\right)^{1+j} \left(\frac{\omega'_g}{\omega'}\right)^{1+j} K(n, n'). \tag{A.7}$$

In deriving equation (A.7) we have used the relation

$$\mathrm{d}n_g = \left(\frac{\omega_g}{\omega}\right)^{-1} \mathrm{d}n.$$

The kernel K is, up to a constant $\kappa(j)$, uniquely determined and is given by

$$K(n, n') = \kappa(j)[n, n']^{-1-j} \tag{A.8}$$

where $[n, n'] = n_0 n'_0 - n_1 n'_1 - n_2 n'_2$. The verification of equation (A.8) is based on the relation

$$[n_g, n'_g] = \left(\frac{\omega_g}{\omega}\right)^{-1} \left(\frac{\omega'_g}{\omega'}\right)^{-1} [n, n'] \tag{A.9}$$

which is obviously a consequence of the relation

$$[\zeta_g, \zeta'_g] = [\zeta, \zeta']$$

where $\zeta_g = \zeta g$, $\zeta'_g = \zeta' g$.

The module of constant κ is fixed by the normalization relation, which gives

$$|\kappa|^2 = \frac{1}{2\pi} \rho \tanh \pi\rho. \tag{A.10}$$

Thus

$$(Sf)(\varphi) = \frac{2^j}{\sqrt{\pi}} \frac{\Gamma(1+j)}{\Gamma(-\frac{1}{2} - j)} c \int_0^{2\pi} \mathrm{d}\varphi' [1 - \cos(\varphi - \varphi')]^{-1-j} f(\varphi') \tag{A.11}$$

where c is the phase factor. (For the sake of brevity, the value of function f at $n = (1, \cos\varphi, \sin\varphi)$ is denoted by $f(\varphi)$.)

Taking into account the functions $|m\rangle = \mathrm{e}^{\mathrm{i}m\varphi}$ forms $SO(2)$ bases in $L^2(S^1)$, we have

$$\langle m'|S|m\rangle = \delta_{mm'} S_m \tag{A.12}$$

where

$$S_m = \frac{1}{2\sqrt{\pi}} \frac{\Gamma(1+j)}{\Gamma(-\frac{1}{2}-j)} c \int_0^{2\pi} \left|\sin\frac{\varphi}{2}\right|^{-2-2j} \mathrm{e}^{-\mathrm{i}m\varphi}\, \mathrm{d}\varphi. \tag{A.13}$$

Using formula 3.829(1) of [17], we obtain from equation (A.13)

$$S_m = c\frac{\Gamma(\frac{1}{2}+\mathrm{i}\rho+m)}{\Gamma(\frac{1}{2}-\mathrm{i}\rho+m)}$$

which, of course, coincides with result (2.12).

(ii) The hyperbolic coordinate system corresponding to the subgroup reduction $SO(2,1) \supset SO(1,1)$ is given by

$$\zeta = w(\cosh\alpha, \sinh\alpha, \tau) \tag{A.14}$$

where $0 \leqslant w < \infty, -\infty < \alpha < \infty$ and

$$\tau = \begin{cases} 1 & \text{if } \zeta_2 > 0 \\ -1 & \text{if } \zeta_2 < 0. \end{cases}$$

Due to the homogeneity condition (A.1), the elementary representations of $SO(2,1)$ can be realized on a space of functions $(f_+(\alpha), f_-(\alpha))$, $f_\tau(\alpha) = F(\cosh\alpha, \sinh\alpha, \tau)$, where the τ-label specifies the sheet. In this realization the operators $U^j(g)$ are given by

$$(U^j(g)f)_\tau(\alpha) = \left(\frac{w_g}{w}\right)^j f_{\tau'}(\alpha_g) \tag{A.15}$$

where w_g, α_g and τ' are determined from the parametrization (A.14) of $\zeta_g = \zeta g$.

Formula (A.15) at $j = -\frac{1}{2} + \mathrm{i}\rho$ gives principal series representations of $SO(2,1)$ on the Hilbert space $L^2(H)$ with inner product

$$(f,g) = \frac{1}{2\pi}\sum_r \int_{-\infty}^{\infty} \overline{f_\tau(\alpha)} g_\tau(\alpha)\, \mathrm{d}\alpha. \tag{A.16}$$

By arguments very similar to those used to obtain (A.11) we can show that the operator S in this realization may be written as

$$(Sf)_\tau(\alpha) = \frac{2^j}{\sqrt{\pi}} \frac{\Gamma(1+j)}{\Gamma(-\frac{1}{2}-j)} c \sum_{\tau'=\pm1} \int_{-\infty}^{\infty} \mathrm{d}\alpha' \,[\cosh(\alpha-\alpha') - \tau\tau']^{-1-j} f_{\tau'}(\alpha'). \tag{A.17}$$

We note that equation (A.17) can be derived from equation (A.11) with the aid of the following relation

$$f(\varphi) = \begin{cases} (\cosh\alpha)^{-j} f_+(\alpha) & \text{if } 0 \leqslant \varphi < \pi \\ (\cosh\alpha)^{-j} f_-(\alpha) & \text{if } \pi \leqslant \varphi < 2\pi \end{cases} \tag{A.18}$$

for $\cos\varphi = \tanh\alpha$.

This then gives the integral representation of the matrix elements of S in $SO(1,1)$ basis. Taking into account that

$$|\mu+\rangle = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\mu\alpha} \\ 0 \end{pmatrix} \qquad |\mu-\rangle = \begin{pmatrix} 0 \\ \mathrm{e}^{\mathrm{i}\mu\alpha} \end{pmatrix} \tag{A.19}$$

we have

$$\langle \mu'\tau'|S|\mu\tau\rangle = \delta(\mu-\mu') S_{\tau\tau'} \tag{A.20}$$

where

$$S_{++} = S_{--} = \frac{2^j}{\sqrt{\pi}} \frac{\Gamma(1+j)}{\Gamma(-\frac{1}{2}-j)} c \int_{-\infty}^{\infty} \mathrm{d}\alpha\, (\cosh\alpha - 1)^{-1-j} \mathrm{e}^{-\mathrm{i}\mu\alpha} \tag{A.21}$$

$$S_{-+} = S_{+-} = \frac{2^j}{\sqrt{\pi}} \frac{\Gamma(1+j)}{\Gamma(-\frac{1}{2}-j)} c \int_{-\infty}^{\infty} \mathrm{d}\alpha\, (\cosh\alpha + 1)^{-1-j} \mathrm{e}^{-\mathrm{i}\mu\alpha}. \tag{A.22}$$

The explicit expression for $S_{\tau\tau'}$ is then (see formulae 3.542(1) of [17])

$$S_{++} = S_{--} = \frac{c}{\pi} \cosh\pi\mu\, \Gamma(\tfrac{1}{2} + \mathrm{i}\rho + \mathrm{i}\mu)\Gamma(\tfrac{1}{2} + \mathrm{i}\rho - \mathrm{i}\mu) \tag{A.23}$$

$$S_{+-} = S_{-+} = -\mathrm{i}\frac{c}{\pi} \sinh\pi\rho\, \Gamma(\tfrac{1}{2} + \mathrm{i}\rho + \mathrm{i}\mu)\Gamma(\tfrac{1}{2} + \mathrm{i}\rho - \mathrm{i}\mu). \tag{A.24}$$

(iii) The parabolic (or horispherical) coordinate system corresponding to the subgroup reduction $SO(2,1) \supset E(1)$ is given by

$$\zeta = W\left(\frac{1+x^2}{2}, \frac{1-x^2}{2}, x\right) \tag{A.25}$$

where $0 \leqslant W < \infty, -\infty < x < \infty$. Consequently principal series of $SO(2,1)$ can be realized on the Hilbert space $L^2(R)$ of square integrable function

$$f(x) \equiv F\left(\frac{1+x^2}{2}, \frac{1-x^2}{2}, x\right)$$

on line R with inner product

$$(f, f') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{f(x)} f'(x)\, \mathrm{d}x. \tag{A.26}$$

It follows from (A.1) and (A.2) that in $L^2(R)$ the operators of the representations take the form

$$(U^j(g)f)(x) = \left(\frac{W_g}{W}\right)^j f(x_g) \tag{A.27}$$

where W_g and x_g are determined from the parametrization (A.25) of $\zeta_g = \zeta g$.

The operator S in this realization is given by

$$(Sf)(x) = \frac{2^j}{\sqrt{\pi}} \frac{\Gamma(1+j)}{\Gamma(-\frac{1}{2}-j)} \int_{-\infty}^{\infty} |x - x'|^{-2-2j} f(x')\, \mathrm{d}x'. \tag{A.28}$$

Note that equation (A.28) can also be derived from equation (A.11) with the aid of the relation

$$f(\varphi) = \left(\frac{2}{1+x^2}\right)^j f(x) \qquad \text{for } \cos\varphi = \frac{1-x^2}{1+x^2}.$$

This then gives the integral representation of the matrix element of S in the $E(1)$ basis. As a result

$$\langle\lambda'|S|\lambda\rangle = \delta(\lambda - \lambda')S_\lambda \tag{A.29}$$

where

$$S_\lambda = \frac{2^j}{\sqrt{\pi}} \frac{\Gamma(1+j)}{\Gamma(-\frac{1}{2}-j)} c \int_{-\infty}^{\infty} \mathrm{d}x\, |x|^{-2-2j} \mathrm{e}^{-\mathrm{i}\lambda x} \tag{A.30}$$

$$= c\lambda^{2\mathrm{i}\rho} \tag{A.31}$$

(see equation 3.761(9) of [17]).

References

[1] For a review see Olshanetsky M A and Perelomov A M 1981 *Phys. Rep.* **71** 14
For a review see Olshanetsky M A and Perelomov A M 1983 *Phys. Rep.* **94** 313
[2] Frank A and Wolf K B 1984 *Phys. Rev. Lett.* **52** 1737
[3] Alhassid Y, Engel J and Wu J 1984 *Phys. Rev. Lett.* **53** 17
[4] Alhassid Y, Gürsey F and Iachello F 1986 *Ann. Phys.* **167** 181
[5] Alhassid Y, Iachello F and Wu J 1986 *Phys. Rev. Lett.* **56** 271
[6] Frank A, Alhassid Y and Iachello F 1986 *Phys. Rev.* A **34** 677
[7] Wu J, Iachello F and Alhassid Y 1987 *Ann. Phys.* **173** 68
[8] Kerimov G A 1998 *Phys. Rev. Lett.* **80** 2976
[9] Knapp A W and Stein E M 1971 *Ann. Math.* **93** 489
Knapp A W and Stein E M 1980 *Inv. Math.* **60** 9
[10] Wong M K F and Yeh H Y 1975 *J. Math. Phys.* **16** 800
[11] For a review see Engelfield M J 1972 *Group Theory and the Coulomb Problem* (New York: Wiley)
[12] Alhassid Y, Gürsey F and Iachello F 1983 *Ann. Phys.* **148** 346
[13] Vilenkin N Ya 1968 *Special Functions and the Theory of Group Representation* (Providence, RI: American Mathematical Society)
[14] Messiah A 1961 *Quantum Mechanics* vol 1 (Amsterdam: North-Holland)
[15] Wehrhahn R F, Smirnov Yu F and Shirokov A M 1992 *J. Math. Phys.* **33** 2384
[16] Dane C and Verdiyev Y A 1996 *J. Math. Phys.* **37** 39
[17] Gradshtyen I S and Ryzhik I M 1980 *Table of Integrals, Series and Products* (New York: Academic)